

Laser Beam Propagation

We'll begin our discussion of beam propagation with a practical topic. The focusing characteristics of a laser beam, which may be of interest to anyone using a laser for work or experimentation.

Focusing characteristics of a real laser beam

Focus spot size due to diffraction: $d_{f0} = \frac{4M^2 \lambda f}{\pi D}$. In this equation, D is the input beam diameter at the lens (Actually I think this is the waist width of the beam before the introduction of the lens) (at $1/e^2$ point), f is the focal length of the lens being used and λ is the wavelength of the laser light. **M^2 is the times diffraction beam quality factor.**

Focus spot size of a real laser beam $d_{f0} = M^2 \frac{4\lambda f}{\pi D}$ is the smallest that the laser beam will focus. The Rayleigh range is the distance over which the irradiance drops by a factor of 2, or the beam size covers twice the area as compared with the waist is given by $Z_r = \frac{\pi w_0^2}{\lambda M^2}$, where M^2 is called the times diffraction limit due to its use in the spot size equation above. The best possible M^2 is unity, all but "perfect" beams have $M^2 > 1$.

Note: The **Lagrange invariant** is $n \cdot 2w_0 \cdot \theta = \frac{4\lambda_0}{K\pi} = \frac{M^2 4\lambda_0}{\pi}$ where n is the index of refraction w_0 is the waist half width and θ is the far field divergence full angle. This "invariant" remains constant through all paths of a beam as long as the optical system is not aperturing and is free of aberrations.

The beam quality factor is a measure of the product of variances of the beams transverse energy distribution and spatial frequency distribution and is constant for any particular beam.

$$M_x^2 = 4\pi \cdot \sigma_{0x} \cdot \sigma_{s_x}$$

The variance of the energy distribution is simply a measure of the beam width while the variance of the spatial frequency distribution is a measure of the number of plane wave components making up the beam and is thus a measure of the angular divergence of the beam.

The far field divergence full angle can be calculated from the *beam diameter at the focus of a lens*, d_f , from:

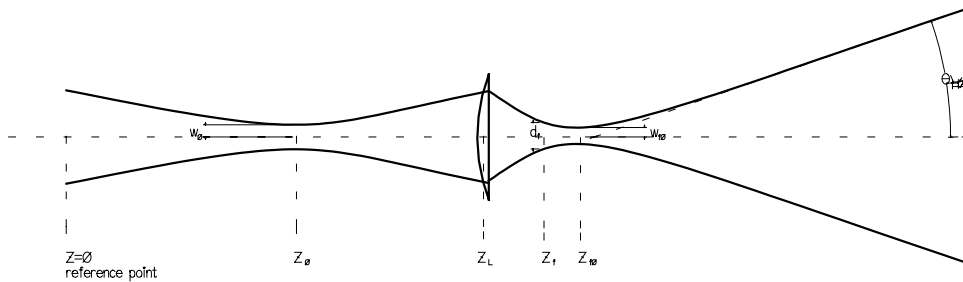
$$\theta = \frac{d_f}{f}, \text{ (Insight into this fact can be gained from Fourier Optics 0 and the Fourier}$$

Transform performed by a lens) so that if $n=1$ the Lagrange invariant leads to:

$$d_f = \frac{M^2 4\lambda f}{\pi 2w_0}, \text{ which says that the diameter of the spot at the focal length of a lens of}$$

focal length f is proportional to $4 M^2$ times the focal length times the wavelength and inversely proportional to π times the diameter of the waist before the lens.

When we focus a laser beam, we have a whole new set of parameters. We have the original beam waist location, beam waist width, Rayleigh range divergence angle etc... but we also have the “artificial” parameters, focused beam waist location, focused waist width, focused divergence etc... which all depend on the location of the lens and the focal length of the lens.



The **beam half width after the lens** is given by the following as long as z is greater than the lens position z_L :

$$w(z) = \sqrt{w_0^2 \left(1 - \frac{z - z_L}{f}\right)^2 + \frac{M^4 \lambda^2}{\pi^2 w_0^2} \left[(z_L - z_0) \left(1 - \frac{z - z_L}{f}\right) + z - z_L \right]^2}$$

Equation 1: Focused Beam Width Propagation.

which can be simplified if we set the reference at the waist location:

$$w(z) = \sqrt{w_0^2 \left(1 - \frac{z - z_L}{f}\right)^2 + \frac{M^4 \lambda^2}{\pi^2 w_0^2} \left[z_L \left(1 - \frac{z - z_L}{f}\right) + z - z_L \right]^2}$$

or it can be simplified more by setting the reference at the lens location:

$$w(z) = \sqrt{w_0^2 \left(1 - \frac{z}{f}\right)^2 + \frac{M^4 \lambda^2}{\pi^2 w_0^2} \left[(-z_0) \left(1 - \frac{z}{f}\right) + z \right]^2}$$

Note also, that if f is infinite, then we get the usual arrangement in the case of no lens at all:

$$w(z) = \sqrt{w_0^2 + \frac{M^4 \lambda^2}{\pi^2 w_0^2} (z - z_0)^2}$$

Equation 2: Natural Beam Width Propagation.

Note that the Rayleigh range is:

$$z_R = \frac{\pi w_0^2}{M^2 \lambda}$$

Equation 3: Rayleigh Range.

$$\text{so that } w(z) = \sqrt{w_0^2 + w_0^2 \left(\frac{z - z_0}{z_R} \right)^2}$$

The half width at the focal length of the lens is given by filling in for z the focal length plus the position of the lens:

$$w_f = w(z = z_L + f) = \frac{M^2 \lambda f}{\pi w_0}$$

which fits with the fact that the divergence angle can be derived from the diameter of the beam at the focal length of the lens.

$$\theta = \frac{d_f}{f}, \text{ and } d_f = \frac{M^2 4 \lambda f}{\pi 2 w_0}, \text{ which gives us: } \theta_{1/2} = \frac{w_0}{z_R}$$

The waist of the focused beam is the point at which the half width is a minimum. But before we can find what the waist width will be, we must find the position of the waist.

The location of the focused waist is derived by taking the derivative of the half width with respect to z and then setting it to zero (to find minima of the half width) and then solving for z :

$$z_{f0} = \frac{\frac{w_0^4 \pi^2 (f + z_L)}{M^4 \lambda^2} + z_0^2 (f + z_L) + z_0 (f^2 - 2z_L^2) + z_L^3 - z_L^2 f}{\frac{w_0^4 \pi^2}{M^4 \lambda^2} + z_0^2 + 2z_0 (f - z_L) - 2z_L f + z_L^2 + f^2}$$

Equation 4: Location of Artificial Waist.

if we set the reference point at the original waist we can simplify somewhat:

$$z_{f0} = \frac{\frac{w_0^4 \pi^2 (f + z_L)}{M^4 \lambda^2} + z_L^3 - z_L^2 f}{\frac{w_0^4 \pi^2}{M^4 \lambda^2} - 2z_L f + z_L^2 + f^2}$$

or we can simplify by setting the lens position to be the zero reference:

$$z_{f0} = \frac{\frac{w_0^4 \pi^2 f}{M^4 \lambda^2} + z_0^2 f + z_0 f^2}{\frac{w_0^4 \pi^2}{M^4 \lambda^2} + z_0^2 + 2z_0 f + f^2}$$

Note that z_0 can be negative in any case.

Of course, we could simplify without making any assumptions about the coordinate system used to get the position of the artificial waist:

$$z_{f0} = \frac{w_0^2 \left(\frac{w_0 \pi}{M^2 \lambda} \right)^2 (f + z_L) + z_L \left[(z_L - z_0)^2 - f \right] + f (z_0^2 + z_0 f)}{w_0^2 \left(\frac{w_0 \pi}{M^2 \lambda} \right)^2 + (z_L - f - z_0)^2}$$

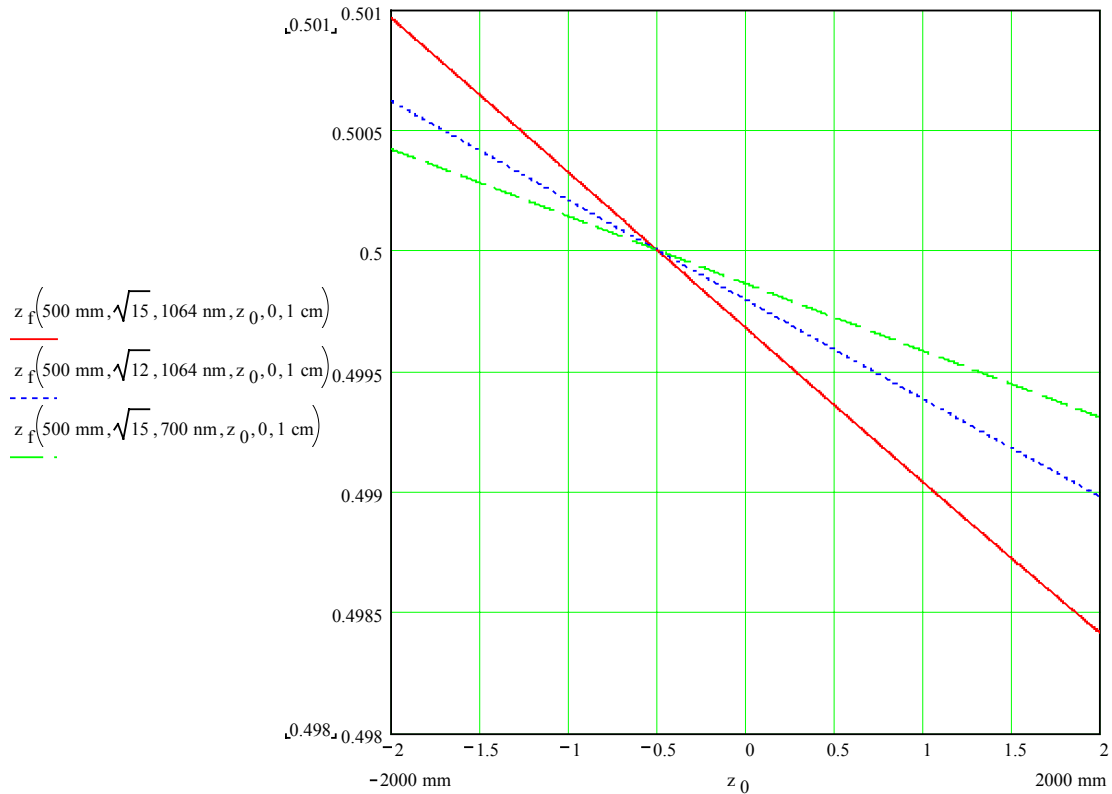


Figure 1 Showing the position of the artificial waist after a lens. Note that the artificial waist is located one focal length beyond the lens only when the original waist is one focal length in front of the lens.

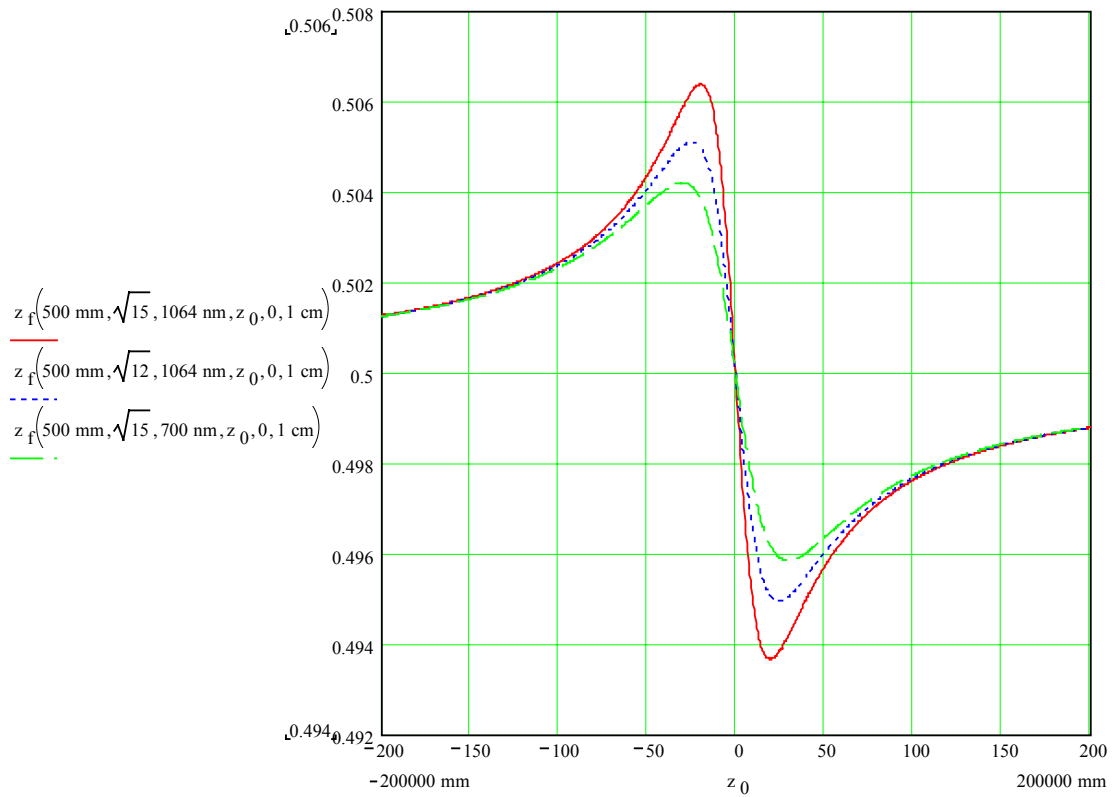


Figure 2 Displacement of the artificial waist away from one focal length beyond the lens goes through a maximum and then asymptotically approaches that position again as we put the original waist farther before or after one focal length before the lens.

The maximum displacement from one focal length after the lens occurs when the original waist is placed at two possible locations measured from the lens location:

$$-f \pm \frac{1}{z_R}$$

or in other words $1/z_R$ on either side of one focal length before the lens.

This maximum displacement of the artificial waist from one focal length beyond the lens has a solution in the reference frame centered on the lens:

$$z_f(z_r, z_{mn}, f) \rightarrow \frac{\left[\frac{f}{zr^2} + \left(-f + \frac{1}{zr}\right)^2 \cdot f + \left(-f + \frac{1}{zr}\right) \cdot f^2 \right]}{\left[\frac{1}{zr^2} + \left(-f + \frac{1}{zr}\right)^2 + \left(-2 \cdot f + \frac{2}{zr}\right) \cdot f + f^2 \right]}$$

or

$$z_f(z_r, z_{mx}, f) \rightarrow \frac{\left[\frac{f}{zr^2} + \left(-f - \frac{1}{zr}\right)^2 \cdot f + \left(-f - \frac{1}{zr}\right) \cdot f^2 \right]}{\left[\frac{1}{zr^2} + \left(-f - \frac{1}{zr}\right)^2 + \left(-2 \cdot f - \frac{2}{zr}\right) \cdot f + f^2 \right]}$$

By putting the position of the artificial waist into the function giving the width of the beam after the lens, we can find the **width of the waist after the lens**:

$$w(z = z_{f0}) = \sqrt{w_0^2 \left(1 - \frac{z_{f0} - z_L}{f}\right)^2 + \frac{M^4 \lambda^2}{\pi^2 w_0^2} \left[(z_L - z_0) \left(1 - \frac{z_{f0} - z_L}{f}\right) + z_{f0} - z_L \right]^2}$$

Which simplifies to:

$$w(z = z_{f0}) = \left(\frac{M^2 \lambda}{\pi w_0} \right) \sqrt{\frac{\left(\frac{w_0 \pi}{M^2 \lambda} \right)^2 w_0^2 f^2 + z_L^2 (1 - z_L)^2}{\left(\frac{w_0 \pi}{M^2 \lambda} \right)^2 w_0^2 + (z_L - f - z_0)^2}}$$

Equation 5: Width of the Artificial Waist.

The graph below shows that the waist size after the lens is affected by several things: The shorter wavelength gives smaller artificial waists for the same size original waist, or we could think of the size of the artificial waist being proportional to the ratio of the wavelength to the original waist size. Shorter focal lengths give smaller waist sizes. Lower M^2 values give smaller waist sizes. And, *note that the maximum waist size is obtained when the original waist is one focal length in front of the lens*. This maximum waist size is the size of the spot at one focal length beyond the lens no matter where the new waist is located. It is this constant that allows us to measure the far field divergence angle of the beam before the lens was introduced.

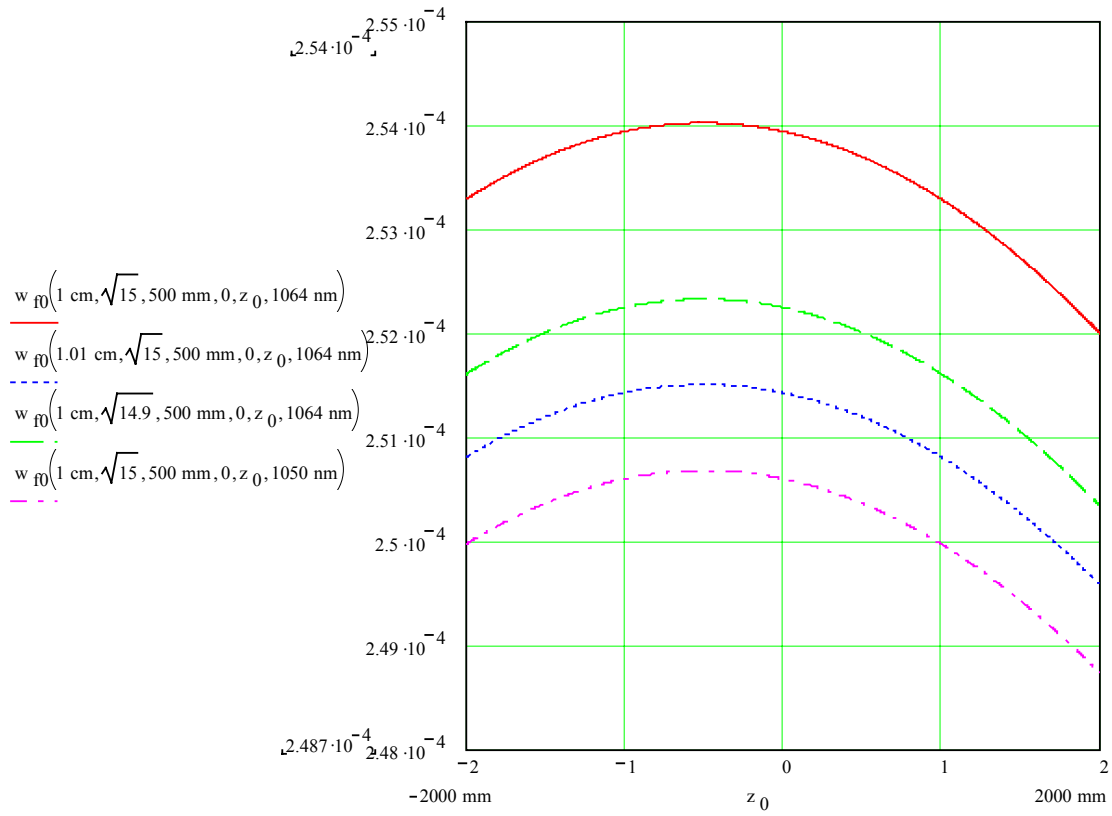


Figure 3 Waist size after the lens

If we step back further in order to look for a minimum waist size we see that the waist size approaches the minimum asymptotically as the original waist is placed infinitely far in front of or beyond the lens.

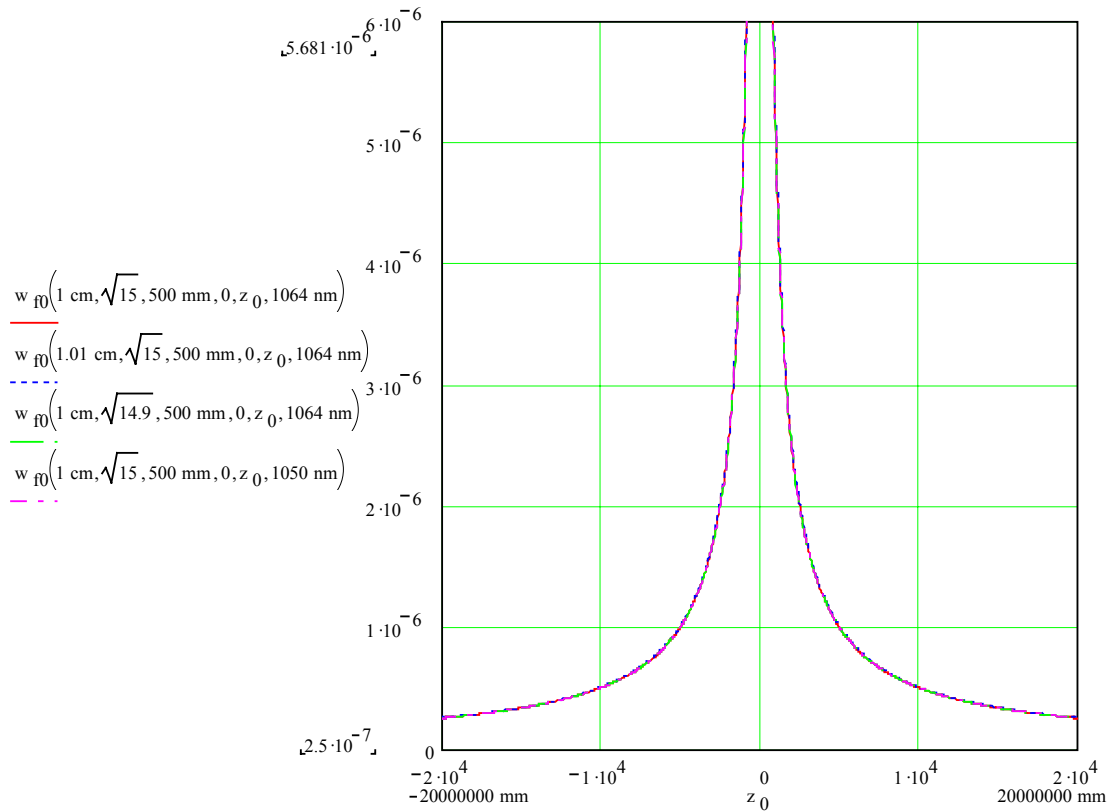


Figure 4 Waist size after the lens again.

Gaussian beam propagation:

With z_0 being the location of the beam waist, z the location of interest along the beam z_r the Rayleigh range w_0 the half width of the beam at the waist and $\theta_{1/2}$ the divergence half angle:

$$w_z^2 = w_0^2 + \theta_{1/2}^2 (z - z_0)^2$$

Equation 6

$$w^2(z) = w_0^2 + M^4 \frac{\lambda^2}{\pi^2 w_0^2} (z - z_0)^2$$

Equation 7

so the divergence half angle is:

$$\theta_{1/2} = M^2 \frac{\lambda}{\pi \cdot w_0}$$

Equation 8

For a Gaussian TEM₀₀ beam the divergence is just:

$$\theta_{1/2} = \frac{\lambda}{\pi \cdot w_0}$$

Equation 9

or

$$w^2(z) = w_0^2 \left[1 + \left(\frac{z - z_0}{z_r} \right)^2 \right]$$

Equation 10

with the Rayleigh range definition absorbing the M^2 term:

$$Z_r = \frac{\pi w_0^2}{\lambda M^2}$$

Equation 11: Rayleigh Range.

For a Gaussian TEM₀₀ beam the Rayleigh range is:

$$Z_r = \frac{\pi w_0^2}{\lambda}$$

Equation 12

Radius of curvature of a Gaussian beam:

$$R(z, z_r) := z \left[1 + \left(\frac{z_r}{z} \right)^2 \right]$$

where z_r is the rayleigh range and z is the distance from the waist.

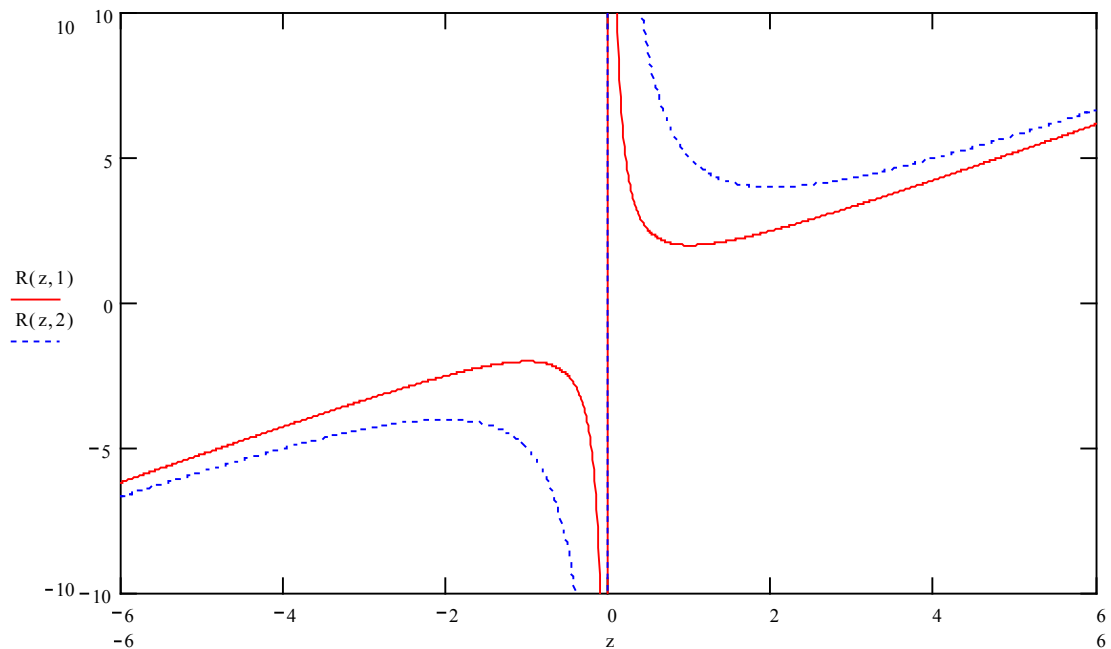


Figure 5 Radius of curvature of a Gaussian beam as a function of the distance from waist.

Radius of curvature is infinite at the waist giving a flat wavefront.

At distances from the waist of exactly the rayleigh range the radius of curvature becomes twice the distance.

$$\frac{d}{dz}R(z, z_r) = 0 \text{ solve, } z \rightarrow \begin{bmatrix} z_r \\ -z_r \end{bmatrix} \quad \text{And that is where the radius of curvature is at a minimum.}$$

Gaussian Beam Formulas

Complex phasor amplitude of ideal TEM₀₀ beam:

$$\tilde{E}(x, y, z) = A_0 \frac{w_0}{w(z)} \exp \left[-\frac{x^2 + y^2}{w^2(z)} - i \cdot \frac{\pi}{\lambda} \frac{x^2 + y^2}{R(z)} - i \frac{2\pi}{\lambda} z + i\zeta(z) \right]$$

Equation 13

$$R(z) = z \cdot \left[1 + \left(\frac{z_r}{z - z_0} \right)^2 \right]$$

$$w(z) = w_0 \cdot \left[1 + \left(\frac{z - z_0}{z_r} \right)^2 \right]^{1/2}$$

$$w_0 = \left(\frac{\lambda \cdot z_r}{\pi} \right)^{1/2} \quad \text{or} \quad z_r = \frac{\pi \cdot w_0^2}{\lambda}$$

$$\zeta(z) = \arctan \left[\frac{z - z_0}{z_r} \right]$$

Equation 14

or

$$\tilde{E}(x, y, z) = \frac{A_1}{z - z_0 + iz_r} \exp[-ik(z - z_0)] \exp \left[-i \frac{k}{2} \frac{x^2 + y^2}{z - z_0 + iz_r} \right]$$

$z_0 = \text{Position_of_waist}$

$z_r = \text{Rayleigh_Range}$

Equation 15

$$\tilde{E}(x, y, z) = \frac{A_1}{z - z_0 + i \frac{\pi w_0^2}{\lambda}} \exp \left[-i \frac{2\pi}{\lambda} (z - z_0) \right] \exp \left[-i \frac{\pi}{\lambda} \frac{x^2 + y^2}{z - z_0 + i \frac{\pi w_0^2}{\lambda}} \right]$$

Equation 16

Intensity profile of the ideal TEM₀₀ beam:

$$I(x, y, z) = I_0 \left[\frac{w_0}{w(z)} \right]^2 \exp \left[-2 \cdot \frac{x^2 + y^2}{w^2(z)} \right]$$

Equation 17

Or

$$I(x, y, z) = \frac{A_1^2}{(z - z_0)^2 + z_r^2} \cdot e^{-k \frac{(x^2 + y^2) z_r}{(z - z_0)^2 + z_r^2}} = A_1^2 \frac{\lambda^2}{\pi^2 w_0^2} \left[\frac{w_0}{w(z)} \right]^2 \exp \left[-2 \frac{(x^2 + y^2)}{w^2(z)} \right]$$

Equation 18

Gaussian Spot size:

$$w^2(z) = w_0^2 + \frac{\lambda^2}{\pi^2 \cdot w_0^2} (z - z_0)^2 = w_0^2 + w_0^2 \frac{(z - z_0)^2}{z_r^2}$$

Equation 19: Gaussian Spot Size Propagation.

Focused beam size using perfect lens and TEM₀₀ beam:

$$d_{f0} = \frac{4\lambda f}{\pi D}$$

Equation 20: Beam diameter at focal plane.

Spatial Frequency Distribution of TEM₀₀ beam:

Intensity profile of a Gaussian beam

Power density of a Gaussian beam: The maximum intensity as a function of the beam's total power P and the beams diameter D is: $I_m(P, D) = \frac{8P}{\pi D^2}$. For pulsed systems the maximum energy density is:

$E_m(E, D) = \frac{8E}{\pi D^2}$., Intensity of a gaussian beam at some radius r from the center of a beam of diam 2w:

$$I(r, P, w) = \frac{2P}{w^2 \pi} e^{-\frac{2r^2}{w^2}}$$

Where r is the transverse radius from the center of the beam's transverse energy profile.

Gaussian form: $f(r, w) = \exp(-2r^2 / w^2)$

Intensity Profile of Various Hermite Gaussian Modes

A laser beam can be represented by its complex amplitude. This complex amplitude is a linear combination of the complex amplitudes of the Hermite Gaussian modes that compose the beam.

$$U(x, y, z) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} A_{l,m} U_{l,m}(x, y, z)$$

The complex amplitudes of the individual modes are:

$$U_{l,m}(x, y, z) = \left[\frac{w_0}{w(z)} \right] \cdot H_l \left(\frac{\sqrt{2} \cdot x}{w(z)} \right) \cdot \exp \left[\frac{-\sqrt{2} \cdot x}{2 \cdot w(z)} \right] \cdot H_m \left(\frac{\sqrt{2} \cdot y}{w(z)} \right) \cdot \exp \left[\frac{-\sqrt{2} \cdot y}{2 \cdot w(z)} \right] \cdot \exp \left[-ikz - ik \frac{x^2 + y^2}{2 \cdot R(z)} + i(l + m + 1) \tan^{-1} \left(\frac{z - z_0}{z_r} \right) \right]$$

The intensity at any point and thus the beam profile at any z location can be found by finding the norm of the complex amplitude:

$$I(x, y, z) = |U(x, y, z)|^2 = U(x, y, z) \cdot U^*(x, y, z)$$

where the star indicates complex conjugation (all +i's go to -i's, note that the result eliminates any influence from the last exponential which is an oscillating term)

The Hermite polynomials in the expression for the complex amplitude are given by the following recursion relation:

$$H_{l+1}(v) = 2 \cdot v \cdot H_l(v) - 2 \cdot l \cdot H_{l-1}(v)$$

and the first few such polynomials are given here.

$$H_0(v) = 1$$

$$H_1(v) = 2v$$

$$H_2(v) = 4v^2 - 2$$

$$H_3(v) = 8v^3 - 12v$$

$$H_4(v) = 16v^4 - 48v^2 + 12$$

$$H_5(v) = 32v^5 - 160v^3 + 120v$$

$$H_6(v) = 64v^6 - 480v^4 + 720v^2 - 120$$

Note that the phase structure of the entire beam, according to this theory, consists of the same phase as each of the component modes which are all the same, namely

$$\phi(x, y, z) = -kz - k \frac{x^2 + y^2}{2R(z)} + (l + m + 1) \cdot \tan^{-1} \left((z - z_0) / z_r \right),$$

in other words the wavefront is rather boring having no structure to indicate the mode mix.

The wavenumber k is equal to $2\pi/\lambda$ where λ is the wavelength of the laser beam. (We'll ignore linewidth or spectral spread). **Note that this differs from the spectral wavenumber which is just $1/\lambda$ in microns.**

The other factors in the equation are:

$$R(z) = (z - z_0) \cdot \left[1 + \left(\frac{z_r}{z - z_0} \right)^2 \right]$$

$$w(z) = w_0 \cdot \left[1 + \left(\frac{z - z_0}{z_r} \right)^2 \right]^{1/2}$$

$$w_0 = \left(\frac{\lambda \cdot z_r}{\pi} \right)^{1/2} \text{ or } z_r = \frac{\pi \cdot w_0^2}{\lambda}$$

Note that the equation for beam width is altered by the beam quality factor in the following manner:

$$w(z) = w_0 \cdot \left[1 + M^4 \left(\frac{z - z_0}{\pi \cdot w_0^2 / \lambda} \right)^2 \right]^{1/2} \text{ or you can adjust the Rayleigh range } z_R = \frac{\pi \cdot w_0^2}{M^2 \lambda} \text{ and get:}$$

$$w(z) = w_0 \cdot \left[1 + \left(\frac{z - z_0}{z_R} \right)^2 \right]^{1/2}$$

In these equations R is the radius of curvature of the wavefront, z-z₀ is the distance along the beam from the waist and z_r is the Rayleigh range

Note: by choosing the wavelength and the Rayleigh range, or by choosing the wavelength and the waist spot size we can determine all of the characteristics of a Gaussian TEM₀₀ beam.

The M² is given by the l,m values of the mode. (Assuming a clean mode. If the beam consists of a mode mix, then the incoherent assumption says that you figure M² for each mode in the mix and add them with weighting according to the intensity in the modes in the mix. Coherent mode mix theory is more complicated).

M² From Hermite Gaussian Mode Mixture

For a beam composed of a mixture of power-normalized H.G. modes $\tilde{u}_{m,n}(x, y, z)$ such that the phasor amplitude is $\tilde{E}(x, y, z) = \sum_{m,n} \tilde{c}_{m,n} \cdot \tilde{u}_{m,n}(x, y, z)$, then the M² or beam quality factors can be given by:

$$M_x^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (2m+1) \cdot |\tilde{c}_{m,n}|^2 \text{ and } M_y^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (2n+1) \cdot |\tilde{c}_{m,n}|^2$$

Derivation of M².

Following will be the derivation of the beam quality factor M², which is essentially derived from the uncertainty principle. It is used to express the relationship between a beams width and its divergence:

$$M^2 = \frac{\pi \cdot \theta \cdot w_0}{2 \cdot \lambda}$$

Write the beam travelling in the z direction as:

$\tilde{E}(x, y, z, t) = \text{Re} \tilde{E}(x, y, z) \exp[i(\omega t - kz)]$ in which $\tilde{E}(x, y, z)$ is the complex amplitude distribution. This distribution can always be expressed as the Fourier transform in the transverse x,y coordinates of a spatial frequency distribution.

$$\tilde{P}(s_x, s_y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}(x, y, z) \exp[i2\pi(s_x x + s_y y)] dx dy$$

&

$$\tilde{E}(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{P}(s_x, s_y, z) \exp[-i2\pi(s_x x + s_y y)] ds_x ds_y$$

s denotes a spatial frequency.

The beams intensity profile in spatial coordinates is given by: $I(x, y, z) = |\tilde{E}(x, y, z)|^2$. While the beam's profile in spatial frequency coordinates is given by $\hat{I}(s_x, s_y, z) = |\tilde{P}(s_x, s_y, z)|^2$. One is the intensity of the beam in transverse profile at some z location, the other is the relative power of the plane wave component of the beam travelling in some direction indicated by s. This relative power profile in spatial frequency coordinates happens to be independent of z, but I have not yet proved that here.

If we had a plane wave:

$$\begin{aligned} \tilde{E}_{pw}(x, y, z) &= \tilde{E}_0 \exp[-i\vec{k} \cdot \vec{r}] = \tilde{E}_0 \exp[-i(k_x x \sin \theta_x) - i(k_y y \sin \theta_y) - ik_z z] \\ &= \tilde{E}_0 \exp[-i2\pi s_x x - i2\pi s_y y - ik_z z] \end{aligned}$$

So each spatial frequency component of a laser beam characterized by s, corresponds to a plane wave component propagating at an angle theta so that: $s_x = \frac{\sin \theta_x}{\lambda}$ etc.

Under paraxial conditions: $\lambda s_x = \theta_x$,

$$k^2 = k_x^2 + k_y^2 + k_z^2,$$

$k_z = \sqrt{k^2 - k_x^2 - k_y^2} \cong k \sqrt{1 - (\lambda s_x)^2 - (\lambda s_y)^2} \approx k - \pi \lambda (s_x^2 + s_y^2)$, up to about 30 degrees spread angle about the beam axis.

Each plane wave component making up the total beam thus has a z direction propagation behavior:

$$\exp[-ik_z z] \approx \exp[-ikz] \exp[i\pi \lambda (s_x^2 + s_y^2) z]$$

so that

$$\tilde{P}(s_x, s_y, z) = \tilde{P}(s_x, s_y, z_0) \exp[i\pi \lambda (s_x^2 + s_y^2)(z - z_0)]$$

So the individual spatial frequency components making up the complex spatial frequency amplitude do not change in amplitude they only rotate in phase angle as they propagate in free space.

Thus the angular intensity profile of the beam is independent of z:

$$\hat{I}(s_x, s_y) \equiv \left| \tilde{P}(s_x, s_y, z) \right|^2$$

SIDELINE: The numerical propagation method is accomplished by:

- 1 From $\tilde{E}(x, y, z)$, calculate $\tilde{P}(s_x, s_y, z)$ using an appropriate fast fourier transform.
- 2 Propagate by multiplying by $\exp[i\pi\lambda(s_x^2 + s_y^2)(z - z_0)]$.
- 3 $\tilde{P}(s_x, s_y, z) \rightarrow \text{Inverse_FFT} \rightarrow \tilde{E}(x, y, z)$

Moments of the Intensity Distribution

Zero order is the normalization, setting the total power equal to 1 makes normalization easier.

$$P_0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x, y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{I}(s_x, s_y) ds_x ds_y = 1$$

The first order is the center of mass (center of energy in this case).

$$\bar{x}(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x I(x, y, z) dx dy \quad \text{center of mass.}$$

$$\bar{s}_x = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_x \hat{I}(s_x, s_y, z) ds_x ds_y \quad \text{Note the lack of z dependence.}$$

$$\text{First moment propagation rule: } \bar{x}(z) = \bar{x}(z_1) + \lambda \bar{s}_x (z - z_1)$$

Second moment is the variance:

$$\sigma_x^2 \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})^2 I(x, y, z) dx dy$$

$$\sigma_{s_x}^2 \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (s_x - \bar{s}_x)^2 \hat{I}(s_x, s_y, z) ds_x ds_y$$

and in the case of the spatial variance this represents the beam diameter:

$$d_x = 4\sigma_x$$

or

$$w_x = 2\sigma_x$$

The spatial variances are functions of z but the spatial frequency variances are not. The spatial frequency variance describes the divergence of the laser since it describes the bandwidth of the spatial frequencies in the beam and thus the limit of the directions of the plane wave components of the beam. The variance of the spatial frequency can be measured directly by use of the Fourier Transform of a lens. The lens focuses the laser beam onto the focal plane and the power in any plane wave component of the beam becomes the energy density at a location on the focal plane. So the spatial frequency variance can be measured by measuring the beam width at the focal plane of a positive lens.

Intensity at the focal plane is proportional to the Fourier Transform:

$$I(x, y)_{focal_plane} = \frac{1}{(\lambda f)^2} \left| \tilde{P} \left(\frac{x}{\lambda f}, \frac{y}{\lambda f}, z = focal_plane \right) \right|^2$$

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Which leads to:

$$\sigma_{s_x} = \frac{w_{xfp}}{2\lambda f}$$

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Second moment propagation rule:

$$\begin{aligned} \sigma_x^2(z) &= \sigma_x^2(z_1) - A_{1x}(z - z_1) + \lambda^2 \sigma_{s_x}^2 (z - z_1)^2 \\ A_x &\equiv \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} s_x \left[\tilde{P}(s_x, z) \frac{\partial \tilde{P}^*(s_x, z)}{\partial s_x} - \tilde{P}^*(s_x, z) \frac{\partial \tilde{P}(s_x, z)}{\partial s_x} \right] ds_x ds_y \\ \sigma_{0x}^2 &= \sigma_x^2(z_1) - A_{1x} / 4\lambda^2 \sigma_{s_x}^2 \\ z_{0x} &= z_1 + \frac{A_x}{2\lambda^2 \sigma_{s_x}^2} \end{aligned}$$

Alternative formulas for spatial frequency moments:

$$\begin{aligned} \sigma_{s_x}^2 &= \left(\frac{1}{2\pi} \right)^2 \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} \left| \frac{\partial \tilde{E}_0(x, y)}{\partial x} \right|^2 dx dy \\ \sigma_{s_x}^2 &= \left(\frac{1}{2\pi} \right)^2 \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} \frac{1}{4I_0(x, y)} \left(\frac{\partial I_0(x, y)}{\partial x} \right)^2 dx dy \end{aligned}$$

Note that spatial frequency moments all involve transverse derivatives of the waist distribution.

Gaussian beam formulas leading to M^2 :

The Beam Quality Factor is given by

$$M_x^2 = 4\pi \cdot \sigma_{0x} \cdot \sigma_{s_x} = \frac{4\pi}{\lambda f} \left(\frac{w_{xfp}}{2} \right) \left(\frac{w_0}{2} \right) = \frac{\pi d_0 d_{fp}}{4\lambda f}$$

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It is the ratio of the beams spatial frequency variance—space variance product compared to the same variance product for a perfect Gaussian beam TEM₀₀.

The ISO Standard gives the times diffraction limit factor as:

$$M_x^2 = \frac{\pi}{\lambda} \cdot \frac{d_{\sigma_x 0} \Theta_{x\sigma}}{4} = \frac{\pi}{\lambda} \cdot \frac{d_{\sigma_x 0} d_{\sigma_x f}}{4f}$$

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ISO Standard for Measuring M^2

The ISO standard is developed as follows:

Any radially symmetric laser beam is described by the waist location, z_0 , the waist diameter, d_0 , and the far field divergence θ .

Beam propagation is described by $d^2(z) = d_0^2 + \theta^2(z - z_0)^2$

This is assuming the second moments of the power density distribution function are used for the definition of beam widths and divergences.

The quality of the beam is described by the beam propagation factor, K , or the times-diffraction-limit factor, M^2 , which can be derived from the basic data.

The Lagrange invariant is: $n \cdot d_0 \cdot \theta = \frac{4\lambda_0}{K\pi} = \frac{M^2 4\lambda_0}{\pi}$ from which we gather $K = \frac{1}{M^2} = \frac{4\lambda}{d_0\theta}$.

Beam diameters are measured as

$$d(z) = 2\sqrt{2}\sigma(z)$$

$$\sigma^2(z) = \frac{\iint r^2 E(r, z) r dr d\phi}{\iint E(r, z) r dr d\phi}$$

$$r = \sqrt{(x - \bar{x})^2 + (y - \bar{y})^2}$$

$$\bar{x} = \frac{\iint x E(x, y, z) dx dy}{\iint E(x, y, z) dx dy}$$

Note that beam widths in x and y are done slightly differently:

$$d_x(z) = 4\sigma_x(z)$$

$$d_y(z) = 4\sigma_y(z)$$

$$\sigma_x^2(z) = \frac{\iint (x - \bar{x})^2 E(x, y, z) dx dy}{\iint E(x, y, z) dx dy}$$

$$\sigma_y^2(z) = \frac{\iint (y - \bar{y})^2 E(x, y, z) dx dy}{\iint E(x, y, z) dx dy}$$

Note that E here stands for the energy and not the electric field. This may be done with intensity instead of energy.

Then we get M^2 from:

$$M^2 = \frac{\pi d_0 \theta}{\lambda 4}$$

Hermite Gaussian Decomposition of a Measured Beam

(Personal notes on physics and optics)

This section of my notes is an attempt to reconstruct a 2-dimensional method from a paper by **Xin Xue, Haiqing Wei, and Andrew G. Kirk (Intensity-based modal decomposition of optical beams in terms of Hermite-Gaussian functions, J. Opt. Soc. Am. A/Vol. 17 No. 6/ June 2000)**

The intent is to apply the methods found in this paper to finding HG modes in beams measured by the Spiricon M²-200 instrument. Of course I need it to be in both the x and y dimensions and so my notes may contain errors in both my understanding of the paper and in my attempt at an extension to two dimensions. I am also applying this to a focused beam. What I expect to find is the HG composition of the focused beam. How this relates to the original beam is left for another section of my notes.

The intensity profile of any laser beam can be described by a superposition of the intensities due to the “natural modes” of the laser cavity:

$$I(x, y; z) = \sum_{m,n} c_{m,n} |u_{m,n}(x, y; z)|^2.$$

The situation is straight forward if the individual natural modes consist of Hermite Gaussian combinations:

$$u_{m,n}(x, y; 0) = \phi_m(x)\phi_n(y)$$

where

$$\phi_m(x) = \left(\frac{2}{\pi w_0} \right)^{1/4} \frac{1}{\sqrt{2^m m!}} H_m \left(\frac{x\sqrt{2}}{w_0} \right) e^{-\frac{x^2}{w_0^2}}$$

This makes it easy *because* the intensity profile is then sufficient to completely determine the decomposition into Hermite Gaussian modes since the Fourier transform of the intensity due to a Hermite Gaussian mode is a Laguerre Gaussian function which functions form a complete orthogonal set.

$$\mathfrak{F}[\phi_m^2(x)\phi_n^2(y)](p, q) = \psi_m(\pi^2 w_{x0}^2 p^2) \psi_n(\pi^2 w_{y0}^2 q^2) = L_m(\pi^2 w_{x0}^2 p^2) L_n(\pi^2 w_{y0}^2 q^2) e^{-\pi^2 (w_{x0}^2 p^2 + w_{y0}^2 q^2)/2}$$

However, suppose instead, each natural mode is a combination of several HG modes:

$$u_{m,n}(x, y; z) = \sum_{l,q} b_{m,n,l,q} \phi_l(x\eta_x(z)) \phi_q(y\eta_y(z)) e^{[-ikz - ikx^2/2R_x(z) -iky^2/2R_y(z) + i(l+1/2)\tan^{-1}(z/z_{Rx}) + i(q+1/2)\tan^{-1}(z/z_{Ry})]}$$

where

$$b_{m,n,l,q} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_{m,n}(x,y;0) \phi_l(x) \phi_q(y) dx dy$$

$$\eta_x(z) = \frac{1}{\sqrt{1 + \left(\frac{z}{z_{Rx}}\right)^2}}$$

$$k = 2\pi / \lambda$$

$$R_x(z) = z + \frac{z_{Rx}^2}{z}$$

$$z_{Rx} = \frac{\pi W_{x0}^2}{\lambda}$$

Here, there is a slight problem. The Rayleigh range should be modified by the beam quality factor:

$$z_{Rx} = \frac{\pi W_{x0}^2}{M^2 \lambda} \text{ but perhaps this should be different for each order of the HG function:}$$

$$z_{Rx} = \frac{\pi W_{x0}^2}{(2l+1)\lambda}$$

This would mean that η and R are also HG order dependent. Unfortunately this would lead to the following function for the Intensity:

$$\begin{aligned} I(x,y,z) &= \sum_{m,n} |a_{m,n}|^2 \left(\sum_{l,q} |b_{m,n,l,q}|^2 |\phi_l(x\eta_{x,l}(z))|^2 |\phi_q(y\eta_{y,q}(z))|^2 \right) \\ &+ \sum_{m,n} |a_{m,n}|^2 \left\{ \sum_{l_1,l_2,q_1,q_2} b_{m,n,l_1,q_1} b_{m,n,l_2,q_2}^* \phi_{l_1}(x\eta_{x,l_1}(z)) \phi_{l_2}^*(x\eta_{x,l_2}(z)) \phi_{q_1}(y\eta_{y,q_1}(z)) \phi_{q_2}^*(y\eta_{y,q_2}(z)) \right\} \\ &\times \exp \left[ikx^2 \left(\frac{1}{2R_{x,l_2}(z)} - \frac{1}{2R_{x,l_1}(z)} \right) + iky^2 \left(\frac{1}{2R_{y,q_2}(z)} - \frac{1}{2R_{y,q_1}(z)} \right) \right] \\ &\times \exp \left[i \left(l_1 + \frac{1}{2} \right) \tan^{-1} \left(\frac{z}{z_{Rx,l_1}} \right) - i \left(l_2 + \frac{1}{2} \right) \tan^{-1} \left(\frac{z}{z_{Rx,l_2}} \right) + i \left(q_1 + \frac{1}{2} \right) \tan^{-1} \left(\frac{z}{z_{Rx,q_1}} \right) - i \left(q_2 + \frac{1}{2} \right) \tan^{-1} \left(\frac{z}{z_{Rx,q_2}} \right) \right] \end{aligned}$$

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The problem with this is that the summation of intensity profiles from different z locations would require that I know the weighting at each x and y location for the different modes---this sort of defeats the purpose of the work here. **Then again, maybe I have not taken into account that the waist size for each mode component may be different! Of course this just makes things worse.**

Now if we allow that the Rayleigh range is not HG order dependent and invoke a change of variable $x \rightarrow \xi = x\eta_x(z)$, $y \rightarrow \zeta = y\eta_y(z)$, we can follow the lead of Xin Xue and write the coordinate scaled intensity profile at some z location as:

$$I(\xi, \zeta, z) = \sum_{m,n} |a_{m,n}|^2 |u_{m,n}(\xi, \zeta, z)|^2$$

$$= \sum_{p,q} c_{l,q} \phi_l(\xi) \phi_q(\zeta) + \sum_{l_1, l_2, q_1, q_2} d_{l_1 l_2 q_1 q_2} \phi_{l_1}(\xi) \phi_{l_2}(\xi) \phi_{q_1}(\zeta) \phi_{q_2}(\zeta)$$

$$\times \exp[i(l_2 - l_1) \tan^{-1}(z/z_{Rx}) + i(q_2 - q_1) \tan^{-1}(z/z_{Ry})]$$

where the coefficients are given by:

$$c_{l,q} = \sum_{m,n} |a_{m,n}|^2 |b_{m,n,l,q}|^2$$

$$d_{l_1 l_2 q_1 q_2} = \sum_{m,n} |a_{m,n}|^2 b_{m,n,l_1,q_1}^* b_{m,n,l_2,q_2}$$

Further simplification could be made if the Rayleigh range did not depend on whether we were looking at the x or y coordinate.

The cross terms representing inter-modal interference with the d coefficients usually won't vanish which is what makes modal analysis from measured intensity profiles so difficult. We could, however, wash out the cross terms if we properly superimpose a number of intensity profiles to get a synthesized intensity profile:

$$J(\xi, \zeta) = \int_{-\infty}^{\infty} [I(\xi, \zeta, z) + I(-\xi, -\zeta, z)] \frac{z_{Rx} + z_{Ry}}{2 \left(z^2 + \left(\frac{z_{Rx} + z_{Ry}}{2} \right)^2 \right)} dz = 2\pi \sum_{m,n} c_{m,n} \phi_m^2(\xi) \phi_n^2(\zeta)$$

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Here, I have used the average Rayleigh range in the place where the Rayleigh range goes in the one-dimensional case. *(I need to justify this approach, just a guess for now).*

I could decide that there is only one Rayleigh range independent of the two transverse dimensions and then the synthesized intensity becomes:

$$J(\xi, \zeta) = \int_{-\infty}^{\infty} [I(\xi, \zeta, z) + I(-\xi, -\zeta, z)] \frac{z_R}{2(z^2 + z_R^2)} dz = 2\pi \sum_{m,n} c_{m,n} \phi_m^2(\xi) \phi_n^2(\zeta)$$

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With this synthesized intensity profile we can determine the modal weights by an overlap integral in the Laguerre---Gaussian functions:

$$c_{n,m} = 2\pi w_{x0}^2 w_{y0}^2 \int_0^{\infty} \int_0^{\infty} \tilde{J}(p, q) \psi_m(\pi^2 w_{x0}^2 p^2) \psi_n(\pi^2 w_{y0}^2 q^2) p q dp dq$$

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Where the Fourier transform of the synthesized profile has been used.

$$\tilde{J}(p, q) = \mathfrak{F}\{J(\xi, \zeta)\}(p, q)$$

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Note that we have used the relationship: $\phi_m(-\xi) = (-1)^m \phi_m(\xi)$. And continuous integration over the z-axis will eliminate even combinations while odd combinations cancel. Unfortunately, integration over the entire z-axis is not practical so we need an approximation.

The approximation we need is:

$$J_N(\xi, \zeta) = \sum_{k=0}^{N-1} [I(\xi, \zeta, z_k) + I(-\xi, -\zeta, z_k)]$$

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Where N is the number of profiles used and the positions of the required profiles are given by:

$$z_k = z_R \tan \theta_k$$

$$\theta_k = \theta_0 + \frac{k\pi}{N}$$

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Is this set of points required to be symmetric about the waist location?

With this approximation, we are limited to finding HG modes of order less than N.

We probably won't get the profiles at exactly the right locations so we may need to interpolate from the profiles we do measure to get the profiles we should have at the correct locations.

In our instrument at Spiricon, we will already have two important parameters: the beam waist location and size. With this information, we know the correct locations for the required profiles. Thanks to the sampling theorem an interpolation method does exist that can reconstruct the profiles at the correct positions from the actually measured profiles.

Using the Wigner Distribution to Extract Spatial Phase and Coherence Properties

Mode Coupling Coefficients

Fourier Optics

The intensity of light falling on a surface can be represented as the square of a supposed real wavefunction:

$$I(\vec{r}, t) = 2\langle u^2(\vec{r}, t) \rangle$$

Where the real wavefunction is the real part of a complex wavefunction:

$$u(\vec{r}, t) = \frac{1}{2} [U(\vec{r}, t) + U^*(\vec{r}, t)]$$

and the complex wavefunction consists of a complex amplitude times the harmonic function in time representing the frequency of the light (monochromatic for now):

$$U(\vec{r}, t) = U(\vec{r})e^{i\omega t}$$

The time independent factor is the complex amplitude and will consist of a harmonic function along the axis of travel times an arbitrary amplitude function (arbitrary as long as it satisfies the Helmholtz equation):

$$U(\vec{r}) = a(\vec{r})e^{i\vec{k}\cdot\vec{r}}$$

or

$$U(\vec{r}) = a(\vec{r})e^{i\phi(\vec{r})}$$

Now lets start with a plane wave with complex amplitude:

$$U(x, y, z) = A \exp[-i(k_x x + k_y y + k_z z)]$$

with the wavevector $\vec{k} = k_x \hat{i} + k_y \hat{j} + k_z \hat{k}$ and $|\vec{k}| = k = \sqrt{k_x^2 + k_y^2 + k_z^2} = 2\pi / \lambda$.

This wavevector makes angles with the y-z and x-z planes:

$$\theta_x = \sin^{-1}(k_x / k)$$

$$\theta_y = \sin^{-1}(k_y / k)$$

The paraxial approximation holds if the z component of k is much larger than either of the other two components of k.

If we cut through this plane wave with constant Z plane and call that location Z=0, then the cross section impinging on this plane is the complex amplitude profile of the plane wave on this (x, y) plane. (Note that the square of this complex amplitude gives the intensity profile on this (x, y) plane and in the case of the plane wave it is uniform illumination).

The complex amplitude in z=0 plane:

$$U(x, y, 0) = f(x, y) = A \exp[-i2\pi(\nu_x x + \nu_y y)]$$

$$\nu_x = k_x / 2\pi$$

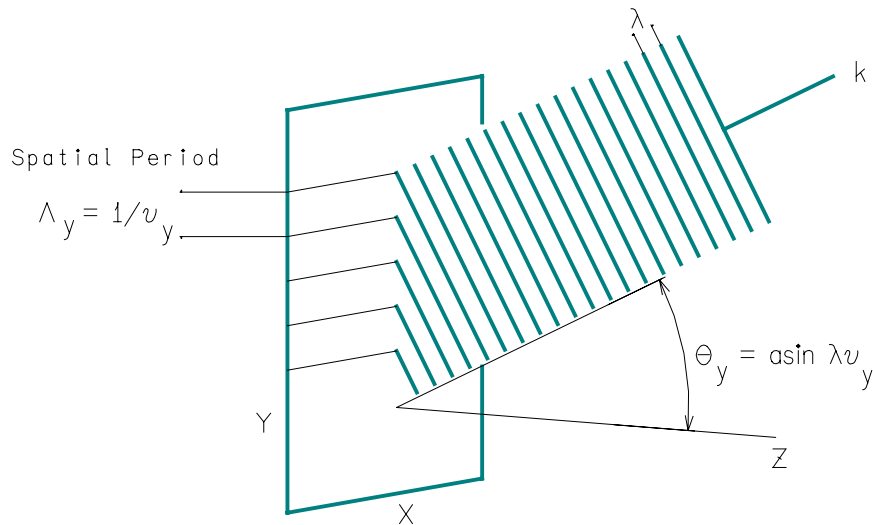
$$\nu_y = k_y / 2\pi$$

Here, ν_x and ν_y are the **spatial frequencies of the complex amplitude** along the x and y directions on the z=0 plane.

Note now that if the paraxial approximation applies, then:

$$\theta_x \approx \lambda v_x$$

$$\theta_y \approx \lambda v_y$$



As a side note a hologram or a diffraction grating function by imposing a complex amplitude transmittance at a certain plane so that the transmitted wave has this function $f(x, y)$ forced on it at that plane. A plane wave in the z direction becomes a plane wave in the z direction plus plane waves at the appropriate angles such that the forced spatial frequencies apply. Since these spatial frequencies, which are now forced by transmission properties, would have depended on both angle and frequency of an incoming wave, the angle of the transmitted wave depends on the temporal frequency of the wave and thus a diffraction grating acts to separate different wavelength components of a beam into beams in different directions.

Now suppose the complex transmittance imposed on a plane wave, or equivalently, the complex amplitude cross section of a light beam, $f(x, y)$, is a superposition of many spatial frequencies (for example a combination of spatial frequencies making up a photograph).

$$f(x, y) = \iint F(v_x, v_y) \exp[-i2\pi(v_x x + v_y y)] d v_x d v_y$$

Where $F(v_x, v_y)$ is the amplitude of the (v_x, v_y) spatial frequency combination. Then the transmitted light (or the light moving beyond this plane in the beam) can be decomposed into a superposition of plane waves such that each combination of spatial frequencies (v_x, v_y) is represented by a plane wave moving in a unique direction. In this view, $F(v_x, v_y)$ is the complex envelope of the plane wave component of the beam moving in that unique direction. The intensity profile of the light on a given plane is the square of this resultant complex amplitude function.

$$I(x, y, z) = |U(x, y, z)|^2$$

$$U(x, y, z) = \iint F(\nu_x, \nu_y) \exp[-i(2\pi\nu_x x + 2\pi\nu_y y)] \exp(-ik_z z) d\nu_x d\nu_y$$

$$= \iint F(\nu_x, \nu_y) \exp(-ik_z z - ik_x x - ik_y y) d\nu_x d\nu_y$$

Note that in the last integral the values of k are dependent on the values of the spatial frequency combination.

Seen in this way, free space propagation of a beam acts as a “spatial prism” separating the spatial frequency components of a beams complex profile just as a prism separates the temporal frequency components.

The function of a lens places the energy of a plane wave with a particular direction at a particular location on the focal plane. Since we have a unique direction for every spatial frequency component of a beam then we would have a unique location on the focal plane of a lens for each spatial frequency component of the beam falling on the lens. The lens maps each direction (θ_x, θ_y) into a single point

$$(x, y) = (\theta_x f, \theta_y f) \approx (\lambda f \nu_x, \lambda f \nu_y).$$

If the complex amplitude of the beam falling in the lens is: $f(x, y)$ then the amplitude of a given spatial frequency pair (ν_x, ν_y) and thus a given direction (θ_x, θ_y) , is the Fourier transform of this complex amplitude:

$$f(x, y) = \iint F(\nu_x, \nu_y) \exp[-i2\pi(\nu_x x + \nu_y y)] d\nu_x d\nu_y$$

$$\Leftrightarrow$$

$$F(\nu_x, \nu_y) = \iint f(x, y) \exp[i2\pi(\nu_x x + \nu_y y)] dx dy$$

Then the complex amplitude of the beam on the focal plane is given by:

$$g(x, y) \propto F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

This means that the complex amplitude on the focal plane is proportional to the Fourier transform of the complex amplitude of the beam falling on the lens. This is a result of the fact that direction of a plane wave component at the lens is transformed to a point on the focal plane of the lens. The power at a point on the focal plane then indicates the power in a plane wave component of a beam at the lens.

If we begin with an input plane a distance d from the lens, then the proportionality factor can be calculated from the transfer function of the free space traveled and the transfer function of the lens. The final result is that the complex amplitude on the back focal plane at the position (x, y) is proportional to the Fourier transform of the complex amplitude on the input plane a distance d in front of the lens evaluated at the frequencies $(x/\lambda f, y/\lambda f)$:

$$g(x, y) = \frac{i}{\lambda f} e^{-i2kf} e^{\left[i\pi \frac{(x^2+y^2)(d-f)}{\lambda f^2} \right]} F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f} \right)$$

$$= \frac{i}{\lambda f} e^{-i2kf} e^{\left[i\pi \frac{(x^2+y^2)(d-f)}{\lambda f^2} \right]} \iint f(x', y') \exp\left[i2\pi \left(\frac{x}{\lambda f} x' + \frac{y}{\lambda f} y' \right) \right] dx' dy'$$

Note that the resultant intensity is independent of the distance d:

$$I(x, y) = \frac{1}{(\lambda f)^2} \left| F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f} \right) \right|^2$$

If we set the distance d equal to the front focal length, we simplify the equation a little further:

$$g(x, y) = \frac{i}{\lambda f} e^{-i2kf} F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f} \right)$$

Now suppose we put a screen at the back focal length, and screen out some specific spatial frequencies, then put another lens one focal length beyond this screen. On the back focal plane of this new lens, we will re-image the first plane one focal length in front of the first lens but we will have filtered out specific spatial frequency combinations. For example, a screen blocking the central portion near the axis of the system will screen out low spatial frequencies and the result is an image outlining the high contrast locations of the original. This is a High Pass filter. A filter that consists of a small hole around the center portion and everything else blocked acts as a Low Pass filter blocking all of the high spatial frequencies and resulting in a softened image compared to the original that will have slower fluctuation over the image plane. This situation corresponds to the pinhole filtering of a laser beam in a beam expansion telescope.

Original Image



High Pass Filtered



Low Pass Filtered



Fourier Propagation of a Beam (Mathematical Formulas)

Given a complex valued input function $f(x, y)$, the square of which would be the intensity of the light on a plane, how do you find the function $g(x, y)$, the square of which would be the intensity of light at

another plane due to propagation from the first plane? The answer is that you use a transfer function, $\mathfrak{T}(v_x, v_y)$, in the following way:

$$F(v_x, v_y) = \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} f(x, y) e^{[i2\pi(v_x x + v_y y)]} dx dy$$

$$g(x, y) = \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} \mathfrak{T}(v_x, v_y) F(v_x, v_y) e^{-[i2\pi(v_x x + v_y y)]} dv_x dv_y$$

The only trick now is to find that transfer function for a particular situation.

Free Space Transfer function

Suppose we have a plane wave at the plane $z = 0$.

$$f(x, y) = U(x, y, 0) = A e^{-i2\pi(v_x x + v_y y)} e^{-ik_z 0}$$

At a later plane $z = d$ we expect to see the plane wave:

$$g(x, y) = U(x, y, d) = A e^{-i2\pi(v_x x + v_y y)} e^{-ik_z d}$$

In order to make the required change in the Fourier transform of $f(x, y)$ so that the inverse Fourier transform results in $g(x, y)$ we need to remember:

$$k_z = (k^2 - k_x^2 - k_y^2)^{1/2} = 2\pi \left(\frac{1}{\lambda^2} - v_x^2 - v_y^2 \right)^{1/2}$$

so that the required transfer function is:

$$\mathfrak{T}(v_x, v_y) = e^{-i2\pi d \left(\frac{1}{\lambda^2} - v_x^2 - v_y^2 \right)^{1/2}}$$

Thus:

$$g(x, y) = \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} e^{-i2\pi d \left(\frac{1}{\lambda^2} - v_x^2 - v_y^2 \right)^{1/2}} F(v_x, v_y) e^{-[i2\pi(v_x x + v_y y)]} dv_x dv_y$$

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Where:

$$F(v_x, v_y) = \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} f(x, y) e^{[i2\pi(v_x x + v_y y)]} dx dy$$

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Or, taking advantage of the series expansion in the exponent:

$$g(x, y) \approx e^{-ikd} \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} e^{i\pi\lambda d (v_x^2 + v_y^2)} F(v_x, v_y) e^{-[i2\pi(v_x x + v_y y)]} dv_x dv_y$$

Equation Error! No text of specified style in document.:34 Fresnel approximation of output plane, free space propagation, frequency space transfer.

There is another approach, one using only the space domain instead of the frequency space domain. This treats the free space propagation as a convolution:

$$g(x, y) \approx \frac{i}{\lambda d} e^{-ikd} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') e^{-i\pi \frac{(x-x')^2 + (y-y')^2}{\lambda d}} dx' dy'$$

Equation Error! No text of specified style in document.:35 Fresnel approximation of output plane, free space propagation, space domain convolution.

If we used this free space transfer function to go from input to output planes far enough apart, then the individual plane wave components of the input plane would contribute to the complex amplitude at individual points in the output plane.

$$g(x, y) \approx \frac{i}{\lambda d} e^{-ikd} F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)$$

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This is the **Fraunhofer Approximation** as is:

$$g(x, y) \approx \frac{i}{\lambda d} e^{-ikd} e^{-i\pi \frac{x^2 + y^2}{\lambda d}} F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)$$

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Thus the free space propagation will separate the Fourier components of the beam. The lens discussed earlier is simply a way to accelerate that process. This is the **far field** result. This approximation is valid as long as the complex function $f(x, y)$, in the input plane $z=0$ is confined to a radius b such that $b^2/\lambda d \ll 1$. and we confine ourselves to points in the output plane within a radius a such that $a^2/\lambda d \ll 1$.

Fresnel Approximation

If the function $f(x, y)$, contains only features much larger than the wavelength of the light used so that $v_x^2 + v_y^2 \ll 1/\lambda^2$, then the paraxial approximation is appropriate so that the angles associated with the plane wave components of the beam make angles $\theta_x \approx \lambda v_x, \theta_y \approx \lambda v_y$. The phase factor in the transfer function can then be expanded in a Taylor series resulting in the **Fresnel approximation**:

$$\mathfrak{S}(v_x, v_y) = e^{-ikd} e^{i\pi\lambda d(v_x^2 + v_y^2)}$$

which is valid as long as the following conditions are observed:

If a is the largest radial distance in the output plane, then:

$$\frac{a^4}{4\lambda d^3} \ll 1 \rightarrow \frac{N_F \theta_m^2}{4} \ll 1$$

where

$$N_F = a^2 / \lambda d, \theta_m \approx a / d$$

Then we use:

$$g(x, y) = \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} \mathfrak{H}(v_x, v_y) F(v_x, v_y) e^{-[i2\pi(v_x x + v_y y)]} dv_x dv_y$$

where

$$F(v_x, v_y) = \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} f(x, y) e^{[i2\pi(v_x x + v_y y)]} dx dy$$

Now that we have this free space transfer function, we can use the properties of the Fourier transform to get the free space transform as a convolution in the Fresnel approximation:

$$g(x, y) = \frac{i}{\lambda d} e^{-ikd} \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} f(x', y') e^{-i\pi \frac{(x-x')^2 + (y-y')^2}{\lambda d}} dx' dy'$$

Note that this approach treats the complex amplitude in the output plane as an expansion of paraboloidal waves originating at the input plane, while the first approach treated the output complex amplitude as an expansion of plane waves originating at the input plane.

Lens Transfer Function

Beginning with a complex amplitude $f(x, y)$ at the input plane and using the transfer function of free space to go from an input plane a distance d in front of a lens and then the transfer function of the lens

$e^{i\pi \frac{(x^2 + y^2)}{\lambda f}}$ and then the free space transfer function again from the lens to the output plane a distance f beyond the lens we get for the complex amplitude at the output plane (focal plane of the lens):

$$g(x, y) = \frac{i}{(\lambda f)} e^{-ik(d+f)} e^{i\pi \frac{(x^2 + y^2)(d-f)}{\lambda f^2}} F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

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$$F(v_x, v_y) = \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} f(x, y) e^{[i2\pi(v_x x + v_y y)]} dx dy$$

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Note that the intensity of the light at the focal plane is equal to the absolute value squared of the output plane complex amplitude:

$$I(x, y) = \frac{1}{(\lambda f)^2} \left| F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \right|^2$$

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The intensity is independent of the distance d . If the distance d is equal to the focal length of the lens f , then the complex amplitude itself is somewhat simplified.

$$g(x, y) = \frac{i}{(\lambda f)} e^{-i2kf} F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

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The intensity in the output plane at (x,y) is proportional to the power in the plane wave component of the input beam with angular directions $(\theta_x, \theta_y)=(x/f, y/f)$. This explains why the divergence of the beam can be measured by the beam width at this output plane divided by the focal length of the lens.

4-f System with a Spatial Filter

If we put an input plane at one focal length in front of a lens, a spatial filter one focal length beyond that lens, a second lens one focal length beyond the spatial filter and finally an output plane one focal length beyond the second lens, then we have a 4-f system with a spatial filter. If we designate the transmittance of the spatial filter as $p(x,y)$, then the transfer function and impulse response function of the whole system are given by:

$$\mathfrak{F}(v_x, v_y) = p(\lambda f v_x, \lambda f v_y)$$

$$h(x, y) = \frac{1}{(\lambda f)^2} P\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

where

$$P(v_x, v_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) e^{i2\pi(v_x x + v_y y)} dx dy$$

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So that the output plane complex amplitude is given by:

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\lambda f v_x, \lambda f v_y) F(v_x, v_y) e^{-i2\pi(v_x x + v_y y)} dv_x dv_y$$

$$F(v_x, v_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{[i2\pi(v_x x + v_y y)]} dx dy$$

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Where $f(x,y)$ is the input plane complex amplitude.

Or we can use the impulse response function:

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') \frac{1}{(\lambda f)^2} P\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) dx' dy'$$

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